Persistent Approximation Property for C*-algebras with propagation

(jointwork with G. Yu)

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June 17, 2013

Special Week on Operator Algebras, June 17–21, 2013,
Research Center for Operator Algebras, East China Normal University.

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Definition

- *If T is an operator of $\mathcal{L}(\mathcal{H})$, then* Supp T *is the complementary of the open subset of $X \times X$*

$$\{(x, y) \in X \times X \text{ such that } \exists f \text{ and } g \in C_c(X) \text{ such that } f(x) \neq 0,\ g(y) \neq 0 \text{ and } \pi(f) \cdot T \cdot \pi(g) = 0\}$$

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- *T has propagation less than r if $d(x, y) \leq r$ for all (x, y) in* Supp T.

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- How can we keep track of the propagation and have homotopy invariance?

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If X be a proper metric space and $\pi : C_0(X) \to \mathcal{L}(\mathcal{H})$ is a representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$, let $0 < \varepsilon < 1/4$ (control) and $r > 0$ (propagation). Then q in $\mathcal{L}(\mathcal{H})$ is an ε-r-projection if $q = q^*$, $\|q^2 - q\| < \varepsilon$ and q has propagation less than r.

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- Hence there exists $\kappa : \operatorname{Sp} q \to \{0, 1\}$ continuous and such that $\kappa(t) = 0$ if $t < 1/2$ and $\kappa(t) = 1$ if $t > 1/2$.
- By continuous functional calculus, $\kappa(q)$ is a projection such that $\|\kappa(q) - q\| < 2\varepsilon$;

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- Choosing $Q = Q_{\varepsilon,r}$ with propagation small enought and approximating $((2P_D^* - 1)(2P_D - 1) + 1)^{1/2} P_D ((2P_D^* - 1)(2P_D - 1) + 1)^{-1/2}$ using a power serie, we can for all $0 < \varepsilon < 1/4$ and $r > 0$, construct a ε-r-projection $q_D^{\varepsilon,r}$ such that

$$\text{Ind } D = [\kappa(q_D^{\varepsilon,r})] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right]$$

  in $K_0(\mathcal{K}(L^2(M)) \cong \mathbb{Z}$.

The framework : Filtered algebras

Definition

A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of linear subspaces:

- $A_r \subset A_{r'}$ if $r \leqslant r'$;
- A_r is closed under involution;
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- The elements of A_r are said to have propagation less than r.

Examples

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- C^*-algebras of groups and cross-products:
  - If Γ is a discrete finitely generated group equipped with a word metric. Set

$$\mathbb{C}[\Gamma]_r=\{x\in\mathbb{C}[\Gamma] \text{ with support in } B(e,r)\}.$$

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    Then $C^*_{red}(\Gamma)$ and $C^*_{max}(\Gamma)$ are filtered by $(\mathbb{C}[\Gamma]_r)_{r>0}$.
  - More generally, if Γ acts on a A by automorphisms, then $A \rtimes_{red} \Gamma$ and $A \rtimes_{max} \Gamma$ are filtered C^*-algebras.

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Quantitative K-(semi)-groups

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- $K_1^{\varepsilon,r}(A)$ is an abelian semi-group for $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r}$;

Quantitative K-(semi)-groups

Define for a unital C^*-algebra A, $r > 0$ and $0 < \varepsilon < 1/4$ the homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,r}(A)$ (recall that $\mathrm{P}_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,r}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,r}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ and $h \in \mathrm{P}_\infty^{\varepsilon,r}(C([0,1],A))$ s.t $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.
- $u \sim v$ if there exists $h \in \mathrm{U}_\infty^{\varepsilon,r}(C([0,1],A)$ s.t $h(0) = u$ and $h(1) = v$.

Definition

1. $K_0^{\varepsilon,r}(A) = \mathrm{P}^{\varepsilon,r}(A)/\sim$ *and* $[p,l]_{\varepsilon,r}$ *is the class of* (p,l) *mod.* $\sim$;
2. $K_1^{\varepsilon,r}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,r}$ *is the class of u mod.* $\sim$.

- $K_0^{\varepsilon,r}(A)$ is an abelian group for $[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,r}$;
- $K_1^{\varepsilon,r}(A)$ is an abelian semi-group for $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r}$;
- if u is a ε-r-unitary, then $[u]_{3\varepsilon,2r} + [u^*]_{3\varepsilon,2r} = [1]_{3\varepsilon,2r}$.

The non-unital case

Lemma

$K_0^{\varepsilon,r}(\mathbb{C}) \stackrel{\cong}{\to} \mathbb{Z}$; $[p, l]_{\varepsilon,r} \mapsto \mathit{rank}\, \kappa(p) - l$; $\quad K_1^{\varepsilon,r}(\mathbb{C}) \cong \{0\}$.

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If A is a non unital filtered C^*-algebra and $\tilde{A}$ the unitarization of A,

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If A and B are filtered C^*-algebras with respect to $(A_r)_{r>0}$ and $(B_r)_{r>0}$, a homomorphism $f: A \to B$ is filtered if $f(A_r) \subset B_r$.

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- $A \hookrightarrow A \otimes \mathcal{K}(H)$; $a \mapsto a \otimes e_{1,1}$ induces $K_*^{\varepsilon,r}(A) \stackrel{\cong}{\to} K_*^{\varepsilon,r}(A \otimes \mathcal{K}(\ell^2(\mathbb{N})))$.

Structure homomorphisms

For any filtered C^*-algebra A, $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$, we have natural structure homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A);\ [p,l]_{\varepsilon,r} \mapsto [\kappa(p)] - [I_l];$
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A);\ [u]_{\varepsilon,r} \mapsto [u];$
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- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A);\ [p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'};$
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For any $\varepsilon \in (0, 1/4)$ and any projection p (resp. unitary u) in A, there exists $r > 0$ and q (resp. v) an ε-r-projection (resp. an ε-r-unitary) of A such that $\kappa(q)$ and p are closed and hence homotopic projections (resp. u et v are homotopic invertibles)

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Consequence

For every $\varepsilon \in (0, 1/4)$ and $y \in K_*(A)$, there exists r and x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$.

Controlled index map

- Recall that if D is an elliptic differential operator on a compact manifold M, then for every $0 < \varepsilon < 1/4$ and $r > 0$, there exists $q_D^{\varepsilon,r}$ an ε-r-projection in $\mathcal{K}(L^2(M))$ s.t. $\operatorname{Ind} D = [\kappa(q_D^{\varepsilon,r})] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right]$;

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- We can define in this way a controlled index $\operatorname{Ind}^{\varepsilon,r} D = [q_D^{\varepsilon,r}, 1]$ in $K_0^{\varepsilon,r}(\mathcal{K}(L^2(M)))$ such that $\operatorname{Ind} D = \iota_0^{\varepsilon,r}(\operatorname{Ind}^{\varepsilon,r} D)$;

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2. *the composition*

$$K_0(X) \longrightarrow K_0^{\varepsilon,r}(\mathcal{K}(L^2(X))) \stackrel{\iota_0^{\varepsilon,r}}{\longrightarrow} K_0(\mathcal{K}(L^2(X))) \cong \mathbb{Z}$$

is the index map.

Behaviour for small propagation

Theorem

Let X be a finite simplicial complex equipped with a metric. Then there exists $0 < \varepsilon_0 < 1/4$ such that the following holds :

For every $0 < \varepsilon < \varepsilon_0$, there exists $r_0 > 0$ such that for any $0 < r < r_0$ then

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- Question: Can we have estimations for r_0?

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$$\mu_{\Gamma,A,*}^{\varepsilon,r,d} : KK_*^{\Gamma}(C_0(P_d(\Gamma)), A) \to K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$$

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- We can state a quantitative Baum-Connes conjecture.
- This quantitative Baum-Connes conjecture is implied by the (usual) Baum-Connes conjecture with coefficients.

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Recall that for every $\varepsilon \in (0, 1/4)$ and $y \in K_*(A)$, there exist r and x in $K_*^{\varepsilon,r}(A)$ s.t $\iota_*^{\varepsilon,r}(x) = y$.

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Recall that for every $\varepsilon \in (0, 1/4)$ and $y \in K_*(A)$, there exist r and x in $K_*^{\varepsilon,r}(A)$ s.t $\iota_*^{\varepsilon,r}(x) = y$. How faithfull this approximation is?

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Lemma

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Definition (Persistent Approximation Property)

For A a filtered C^*-algebra and positive numbers ε, ε', r and r' such that $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$, define :

$\mathcal{PA}_*(A, \varepsilon, \varepsilon', r, r')$: for any $x \in K_*^{\varepsilon,r}(A)$, then $\iota_*^{\varepsilon,r}(x) = 0$ in $K_*(A)$ implies that $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0$ in $K_*^{\varepsilon',r'}(A)$.

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Example

If $A=\mathcal{K}(\ell^2(\Sigma))$ for Σ discrete metric set.

- $\mathcal{PA}_0(A,\varepsilon,\varepsilon',r,r')$ holds if for any ε-r-projections q and q' in $\mathcal{K}(\ell^2(\Sigma)\otimes\mathcal{H})$ such that rang $\kappa(q)=$ rang $\kappa(q')$, then q and q' are homotopic ε'-r'-projections up to stabilization.
- $\mathcal{PA}_1(A,\varepsilon,\varepsilon',r,r')$ holds if for any two ε-r-unitaries (in $\mathcal{K}(\ell^2(\Sigma)\otimes\mathcal{H})+\mathbb{C}Id$) are homotopic as ε'-r'-unitaries.

Examples

Definition (Universal example for proper actions)

A locally compact space Z is a universal example for proper actions of Γ if for any locally compact space X provided with a proper action of Γ, there exists $f : X \to Z$ continuous and equivariant, and any two such maps are equivariantly homotopic.

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Examples: Γ hyperbolic, Γ Haagerup with cocompact universal example, Γ fundamental group of a compact oriented 3-manifolds.

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Let Γ be a finitely generated discrete group. Assume that

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- This statement is purely geometric.

Coarse geometry

Let (Σ, d) be a proper discrete metric space;

- Σ has bounded geometry if for all $r > 0$, there exists an integer N such that any ball of radius r has cardinal less than N (example Γ finitely generated group equipped with the word metric) ;

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  - $\forall r > 0, \exists s > 0$ such that $d(x, y) < r \Rightarrow d'(f(x), f(y)) < s$;
- A coarse map $f : \Sigma \to \Sigma'$ is a coarse equivalence if there is a coarse map $g : \Sigma' \to \Sigma$ and $M > 0$ such that $d(f \circ g(y), y) < M$ and $d(g \circ f(x), x) < M \quad \forall x \in X$ and $\forall y \in Y$.

The geometrical Persistent Approximation Property

Definition

Let (Σ, d) a proper discrete metric space. We say that Σ satisfies the geometrical Persistent Approximation Property if there exists $\lambda > 1$ such that for any $0 < \varepsilon \leqslant \frac{1}{4\lambda}$ and any $r > 0$, there exists $r' > r$ and $\varepsilon' \in [\varepsilon, 1/4)$ such that $\mathcal{PA}_*(A \otimes \mathcal{K}(\ell^2(\Sigma)), \varepsilon, \varepsilon', r, r')$ holds for any C^*-algebra A.

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Remark

The geometrical Persistent Approximation Property is invariant under coarse equivalence.

Example

If Γ (finitely generated) satisfies the Baum-Connes conjecture with coefficients and admits a cocompact universal example for proper action, then $|\Gamma|$ satisfies the geometrical Persistent Approximation Property.

Coarse uniform contractibility property

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Example : Σ Gromov hyperbolic.

Coarse embedding in a Hilbert space

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Σ coarsely embeds in a Hilbert space $\mathcal{H}$ if there exists $f : \Sigma \to \mathcal{H}$ s.t : for all $R > 0$, there exists $S > 0$ s.t $d(x, y) < R \Rightarrow \|f(x) - f(y)\| < S$ and $\|f(x) - f(y)\| < R \Rightarrow d(x, y) < S$.

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Theorem

Let Σ be a discrete metric space with bounded geometry. Assume that

- *Σ coarsely embeds in a Hilbert space;*
- *Σ satisfies the uniform coarse contractibility property.*

Then Σ satisfies the geometrical Persistent Approximation Property.

Control indices with coefficients

- Let X be a cpct metric space,and let A be a C^*-algebra. Then for any $0 < \varepsilon < 1/4$ and any $r > 0$, the control index map admits a version with coefficient

$$\mathrm{Ind}_{X,A}^{\varepsilon,r} : KK_*(C(X), A) \to K_*^{\varepsilon,r}(\mathcal{K}(A \otimes L^2(X)))$$

compatible with the maps $\iota_*^{\varepsilon,\varepsilon',r,r'}$ and such that the composition

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- If Σ be a be discrete metric space with bounded geometry and X a compact subset of $P_d(\Sigma)$, then we have an inclusion $\mathcal{K}(L^2(X)) \hookrightarrow \mathcal{K}(L^2(P_d(\Sigma)))$ with propagation increasing controlled independantly on X (indeed $r \mapsto r + cst$).

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- Σ and $P_d(\Sigma)$ are coarse equivalent, hence there is an isomorphism $\mathcal{K}(L^2(P_d(\Sigma))) \cong \mathcal{K}(\ell^2(\Sigma))$ with propagation increasing controlled.

Compactly supported coarse assembly maps

Let Σ be a discrete metric space with bounded geometry, let A be a C^*-algebra. Let us set $A_\Sigma = A \otimes \mathcal{K}(\ell^2(\Sigma))$.

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$$\mu_{\Sigma,A,*}^{\varepsilon,r,s} ; \lim_{X} KK_*(C(X),A) \longrightarrow K_*^{\varepsilon,r}(A_\Sigma),$$

  where in the limit, X runs through compact subspace of the Rips complex $P_s(\Sigma)$, for $r \geqslant r_{\Sigma,d,\varepsilon}$, with $r_{\Sigma,d,\varepsilon}$ decreasing in ε and increasing in d;

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- These maps are compatible with inclusions $P_s(\Sigma) \hookrightarrow P_{s'}(\Sigma)$ and structure map $\iota_*^{\varepsilon,\varepsilon',r,r'} : K_*^{\varepsilon,r}(A_\Sigma) \to K_*^{\varepsilon',r'}(A_\Sigma)$.

Quantitative statements

For Σ discrete metric space with bounded geometry, and A a C^*-alg., we set $K_*(P_d(\Sigma), A) = \lim_{X \subset P_d(\Sigma) \text{ cpct}} KK_*(C(X), A)$.

Quantitative statements

For Σ discrete metric space with bounded geometry, and A a C^*-alg., we set $K_*(P_d(\Sigma), A) = \lim_{X \subset P_d(\Sigma) \text{ cpct}} KK_*(C(X), A)$. Consider

$QI_{\Sigma,A,*}(d, d', r, \varepsilon)$ for any element x in $K_*(P_d(\Sigma), A)$, then $\mu^{\varepsilon,r,d}_{\Sigma,A,*}(x) = 0$ in $K^{\varepsilon,r}_*(A_\Sigma)$ implies that $q^*_{d,d'}(x) = 0$ in $K_*(P_{d'}(\Sigma), A)$.

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Theorem

If Σ uniformaly embeds into a Hilbert space .

1. *Then for any positive numbers d, ε and $r \geqslant r_{\Sigma,d,\varepsilon}$ with $\varepsilon < 1/4$ and $r \geqslant r_{\Sigma,d,\varepsilon}$, there exists a positive number d' with $d' \geqslant d$ such that $QI_{\Sigma,,A}(d, d', r, \varepsilon)$ is satisfied for every C^*-algebra A;*
2. *For some $\lambda > 1$ and for any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{\Sigma,d,\varepsilon} \leqslant r$ and $r' \leqslant r$ such that $QS_{\Sigma,A}(d, r, r', \lambda\varepsilon, \varepsilon)$ is satisfied for every C^*-algebra A.*

Application to Novikov conjecture

Theorem

Let Σ be a be discrete metric space with bounded geometry. Assume that the following assertions hold:

1. *for any d, ε and $r \geqslant r_{\Sigma,d,\varepsilon}$ with $\varepsilon < 1/4$ and $r \geqslant r_{\Sigma,d,\varepsilon}$, there exists d' with $d' \geqslant d$ such that $QI_{F,\mathbb{C}}(d, d', r, \varepsilon)$ is satisfied for any finite subset F of Σ;*
2. *For any ε and r' with $\varepsilon < \frac{1}{4}$, there exist positive numbers d, ε' and r with $r_{\Sigma,d,\varepsilon} \leqslant r$, $r' \leqslant r$ and $\varepsilon < \varepsilon' < \frac{1}{4}$, such that $QS_{F,\mathbb{C}}(d, r, r', \varepsilon', \varepsilon)$ is satisfied for every for any finite subset F of Σ;*

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In particular, if $\Sigma = \Gamma$ is a finitely generated group, then and under the assumptions of the theorem, Γ satisfies the Novikov conjecture.